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Complex Geometry, Unification, and Quantum Gravity. I. The Geometry of Elementary Particles

Thomas R. Love[1]

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The Poincaré group is replaced by $U(3, 2)$, the pseudounitary extension of the de Sitter group $SO(3, 2)$, as internal and space-time symmetries are combined in a geometric setting which invalidates the no-go theorems. A new model of elementary particles as vertical vectors on the principal fiber bundle $U(3, 2) \to U(3, 2)/U(3, 1) \times U(1)$ is introduced and their interactions via Lie bracket analyzed. The model accounts for the four known superselection rules: spin, electric charge, baryon number, and lepton number.

1. INTRODUCTION

For some time, many physicists have felt that a realization of Einstein's vision of a totally unified field theory would be necessary to provide a framework for elementary particle phenomena. The so-called "grand unified theories" (GUTs) attempted to explain the strong and weak nuclear forces and the electromagnetic force in terms of one fundamental force. The most successful of these theories was that of Georgi and Glashow (1974), based on the gauge theory of $SU(5)$. In spite of the initial success of this program, there were several fundamental problems. The GUTs do not attempt to describe gravitation and thus fall short of being totally unified theories. Also, most GUT-type theories lead to the possibility of proton decay and predict particles which have not been observed. The introduction of supersymmetries and "supergravity," designed to incorporate gravity and by-pass the "no-go" theorems, have only introduced more unobserved particles and thus widened the chasm between theory and observation. Consequently, a new program of particle interactions and quantization is needed and one will be introduced in this series of papers.

[1]Department of Mathematical Sciences, University of Nevada-Las Vegas, Las Vegas, Nevada 89154.

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The history of theoretical physics may be viewed as a search for conservation laws. Although the program had to be modified as research progressed, the original motivation was the observation that, within the Lagrangian formalism, conservation laws in turn imply the action of a continuous (Lie) group. Consequently, it seems that the search for a totally unified field theory is a search for the correct Lie group. The Lie group will characterize the conservation laws, and consequently, the geometry of the Lie group will characterize the interactions of the elementary particles as well. There are no conservation laws associated with the so-called "supersymmetries," so they will not be used in the present work.

We begin a study of the elementary particles by describing their interactions, and conversely we begin a study of the fundamental forces by discovering those particles which interact via those forces. Any mathematical model of reality must begin with some assumptions about the way nature works. If we are to base our physics on group theory, it must lead to exact conservation laws: energy, momentum, angular momentum, spin, charge, baryon number, lepton number, etc.

The triumph of general relativity coupled with Einstein's vision of a unified field theory, as well as the successes of the geometric approach to gauge theories, suggest that we should look for a model which includes gravitation (which means that it should be based on geometry, like general relativity), in which the forces are distinct, but ultimately are derived from the same geometry. The successes of gauge theories and of geometric quantization suggest that this geometry of elementary particle interactions should be based on the homogeneous space of some Lie group.

The geometry of elementary particles to be studied here has evolved from the geometric setting the author introduced several years ago (Love, 1984). The many reasons for looking at SU(3, 2) were discussed in that paper, but here the group $SU(3, 2)$ is viewed as an extension of the well-studied de Sitter group $SO(3, 2)$.

Once a group is chosen, there are many ways to obtain physics from the group. The prescription of the gauge theories runs into problems in the case of noncompact groups. But on a more fundamental level, a fatal flaw of the Lagrangian approach is the inevitable appearance of infinities when such a theory is quantized.

The Hamiltonian approach is essentially equivalent to the Lagrangian approach and will not be used as our starting point, although a "Hamiltonian" will appear in the final formulation of the theory. The standard way of introducing groups into physics is to look for the symmetries of the Lagrangian, the Hamiltonian, the S-matrix, or the space-time. The approach taken here is that the group $SU(3, 2)$ is the fundamental object and demands that the physically relevant operators (the observables in quantum theory) and the space-time be constructed from the generators of $su(3, 2)$. We will not be using $SU(3, 2)$ in any standard way. Perhaps the closest concepts in the literature are the dynamical groups of Barut (1980) and Roman (1980) and the spectrum generating groups of Bohm (1986).

No matter how well (or how poorly) motivated the introduction of a specific group is, the most crucial test of the criteria is the agreement of the model developed from the group geometry with experimental data. However, it should be remembered that the pre-Copernican model of the solar system based on epicycles was very precise and very accurate simply because so many people had worked so long refining it. Thus, accurate predictions are not the only criterion for judging a scientific theory. Beyond offering a new level of precision, the present theory has an elegance which could never be found in the perturbations and renormalizations of the standard model.

The problem addressed here was posed by Robert Hermann (1980): "What the Einstein theory has, and elementary particle theory so far lacks, is a compelling geometric foundation and intuition." The work presented here is designed to provide such a geometric foundation for elementary particle theory. The general mathematical setting was also suggested by Hermann (1977): "there are complicated and immensely rich (but badly understood) relations between classical and quantum mechanics, Lie group theory and symplectic manifold theory."

The de Sitter group is a well-studied alternative to the Poincaré group. In a way, $SU(3, 2)$ is a complex version of $SO(3, 2)$. Thus, I am asserting that a complex extension of $so(3, 2)$ is correct instead of a "super-extension" of the Poincaré algebra. The complex structure is a reasonable alternative to the "super-structure" since both assert a discrete symmetry of the algebra. Actually, if complex scalars are permitted, as they must be for quantum theory, then a complex structure and a super structure are identical. A complex structure on the tangent space of a manifold is a mapping $J: TM \to TM$ with $J^2 = -1$. If J is a complex structure, then iJ is a super-structure, for $(iJ)^2 = 1$.

The "no-go" theorems show the impossibility of obtaining the correct mass spectrum of elementary particles with an extension of the Poincaré group by a compact Lie group. These "no-go" theorems show that such an extension is physically irrelevant. The compact symmetries are necessary to describe the strong and weak nuclear forces and to obtain the correct mass spectrum. The Lorentz group is necessary to obtain relativistic dynamics. Since the compact groups cannot be adjoined to the Poincaré group, we must conclude that the Poincaré group is not the physically relevant extension of the Lorentz group. The "no-go" theorems were misnamed; they should have been labeled as the "No way, Poincaré" theorems. Because the Lie algebra used here (the space of vertical vector fields) is infinite dimensional, the no-way Poincaré theorems are not valid (Coleman, 1967).

2. THE ALGEBRA

When trying to use group theory to close the great chasm between quantum theory (the theory of elementary particles) and general relativity, one must learn from the experiences of both sets of scholars. For the relativist, the importance of the group setting lies in the fact that the primary symmetry group G has a subgroup H such that the quotient space G/H yields a variable model for space-time (Halpern, 1983): Minkowski space is the quotient of the Poincaré group modulo the Lorentz group; de Sitter space is $SO(4, 1)/SO(3, 1)$ and "anti-de Sitter space" is $SO(3, 2)/SO(3, 1)$. Taking a clue from these cases, the homogeneous space $SU(3, 2)/SU(3, 1)$ perhaps should be important. But this is a nine-dimensional manifold and the physical relevance is not clear. However, the homogeneous space $SU(3, 2)/SU(3, 1) \times U(1)$ is an eight-dimensional manifold which can be understood physically as a four-complex-dimensional space-time naturally equipped with a pseudo-Hermitian metric of signature $(- - - +)$. Since this homogeneous space-time is a complexification or "quantization" (Rosen, 1962) of $SO(3, 2)/SO(3, 1)$, anti-de Sitter space (AdS), I will dub it "quantum anti-de Sitter space"—QAdS for short. Since QAdS is the quantum version of AdS, it is a physically viable candidate for space-time since the natural metric on QAdS satisfies the complex version of Einstein's equations (Pleblanski, 1975).

For the particle physicist, the important group is the fiber group; the geometry of the fiber bundle determines the "internal structure" of the particles (with the details depending upon the particular model). The research done by elementary particle theorists has shown that $SU(3) \times SU(2) \times U(1)$ is an important group, but the $SU(2)$ factor must be broken. By design, $SU(3) \times SU(2) \times U(1)$ is the maximal compact subgroup of $SU(3, 2)$. But in the passage to the homogeneous space, the symmetry of the group $SU(3, 2)$ is broken to $SU(3, 1) \times U(1)$, whose maximal compact subgroup is $SU(3) \times U(1) \times U(1)$. Thus the only broken compact symmetry is $SU(2)$ breaking to $U(1)$ as required. These observations make some connection with standard quantum theory and are important clues that we are on the right track. The remainder of this series of papers will show that the clues are correct and that the bundle of vertical vector fields forms a viable model of elementary particles.

Having the group and a complex space-time, we are still in a quandary since the standard gauge theory recipes do not work for a noncompact group. The existing recipes for noncompact groups lead to nonrenormalizable theories. To build a workable model requires reworking of the mathematical foundations of quantum theory and relativity. Lie derivatives are used instead of the covariant derivative and "gauge fields" are interpreted as "vertical vector fields" instead of "Lie algebra-valued forms"; thus the present theory is not a "gauge theory."

Mathematically, this is justified by the fact that the Lie derivatives are extensions of the adjoint representation and it is the group action which leads to conservation laws. Thus, each generator of $SU(3, 2)$ has a corresponding Lie derivative and in a Lagrangian theory would have a corresponding conservation law. There is no correspondence between covariant derivatives and conservation laws. The Lie derivatives also seem better suited for dealing with the coherent state formalism (Kaiser, 1990) than the covariant derivative. Physically, the only justification for a new mathematical model is that it explains the data better than does the existing models; that the present model satisfies this criterion will be shown in this series of papers.

The complex space time causes some consternation. If we think of a complex variable as $a+ib$, it seems that we are adding four unobserved dimensions to space-time. From the viewpoint of Kaluza–Klein theories, it is necessary that the extra dimensions be compactified. But to pass from the Lie algebra to the space-time coordinates requires that we exponentiate the $a+ib$ to obtain $e^{a+ib}=e^a e^{ib}$, and hence the additional coordinate e^{ib} is a circle and compact. This coordinate looks more like a phase angle than any other known property of wavefunctions. The complex space-time QAdS can locally be coordinatized as a product of AdS with four circles, reminiscent of the spaces used in some supersymmetry theories. To relate the geometry to physics, the structure constants must be changed to obtain an isomorphic algebra. The size of these circles is governed by the scaling of the structure constants and will determine the strength of the interactions are possible, but does not predict the force strengths.

Fubini *et al.* (1973) (FHJ) suggested a new formulation of quantum field theory based on group theory. They defined three types of generators: kinematic operators, dynamical operators, and the evolution operator. In the present picture we naturally have the same three types of operators: the kinematic operators correspond to the vertical vectors, the dynamical operators correspond to the horizontal vectors. But for the evolution operator to correspond to the dilation operator as it does for Fubini *et al.* requires that we enlarge the group to $U(3, 2)$, where the dilation operator is then the center of $U(3, 2)$. But this does not change the geometry of the base space since $SU(3, 2)/SU(3, 1) \times U(1)$ is diffeomorphic to $U(3, 2)/U(3, 1) \times U(1)$. Thus, a program similar to the FHJ formulation of quantum field theory can be carried out in this non-Euclidean geometric setting.

The introduction of the homogeneous space QAdS is essentially breaking the symmetry from $U(3, 2)$ to $U(3, 1) \times U(1)$. In the Higgs approach to symmetry breaking, each generator of $U(3, 1) \times U(1)$ should then represent a Higgs boson. In the standard model, each of the eight generators of $SU(3)$ represents a gluon, the three generators of $SU(2)$ each represent an intermediate boson, and the $U(1)$ generator is a photon. In the $U(3, 2)$ model, we modify the Higgs program to require that each generator of the broken symmetry represent a family of particles, each of which contains a fundamental particle.

Kursunoglu (1979) observes that the neutron is in some sense a tightly bound state of a proton and a pion: $n = (p, \pi^-)$. Furthermore, it seems that a pion is, in turn, a tightly bound state of an electron and an antineutrino: $\pi^- = (e^-, \bar{\nu})$. Given the importance of Lie algebras in elementary particle physics (Georgi, 1982), a natural question arises: is it possible to replace the above parentheses by Lie brackets?

The answer is not obvious, since there are some fundamental questions to answer. Since $[\pi^-, p] = -[p, \pi^-]$, is $n = [\pi^-, p]$ or is $n = -[\pi^-, p]$? Because of complexities like this, we cannot just write down some relationships and expect them to form a Lie algebra. We must use another plan of attack.

The basic strategy was suggested by Segal (1963):

> . . . after a tentative fundamental symmetry group (or, equivalently a Lie algebra) has been selected, the main steps involved in formulating a specific physical theory of elementary particles may be outlined as follows.
>
> First, certain linear representations of the group must be specified and connected with designated elementary particles (where "elementary" does not necessarily have any absolute meaning, but refers only to its empirically observed role). Second, a maximal Abelian diagonalizable subalgebra of the group algebra must be designated; the spectral values for the elements of the subalgebra give the so-called "quantum numbers" for the particles in question. Usually it is the infinitesimal group algebra or so-called enveloping algebra of the Lie algebra which is employed, after augmentation by the quite limited subgroups of elements in the absolute center of the group, involving only one nontrivial element in the relativistic case, which specifies whether the spin is integral or half integral. Thirdly, these quantum numbers must be connected with experimentally measurable quantities, which involves the construction of a dictionary between the quantum numbers and conventional ones employed with the standard relativistic theory as augmented by various internal quantum numbers such as strangeness, baryon number, etc.

As Segal noted, within the theory of Lie algebras, there is a "natural" spectrum-generating subalgebra: the Cartan subalgebra. The eigenvalue of the spectrum-generating algebra are additive just like the quantum numbers of elementary particles, if their interaction is via Lie bracket. Denote an element of the Cartan subalgebra by γ, and let x and y be eigenvectors of γ; then if $[\gamma, x]=ax$ and $[\gamma, y]=by$, then

$$\begin{aligned}[\gamma,[x,y]]&=[[\gamma,x],y]+[x,[\gamma,y]]\\&=[ax,y]+[x,by]\\&=a[x,y]+b[x,y]\\&=(a+b)[x,y]\end{aligned}$$

When we replace the matrix representation of the Lie algebra by a representation with differential operators and use the Lie derivative, then the same eigenvalues are obtained from the tensor product.

If p is the proton with charge ± 1 and γ is the charge generator, then we should have $[\gamma, p]=p$. If then we take the transpose (or the adjoint), we have

$$[\gamma,p]^{\mathrm{T}}=p^{\mathrm{T}}$$
$$[p^{\mathrm{T}},\gamma^{\mathrm{T}}]=p^{\mathrm{T}}$$

so that $[\gamma^{\mathrm{T}}, p^{\mathrm{T}}]=-p^{\mathrm{T}}$, but we can take γ to be diagonal; then $\gamma^{\mathrm{T}}=\gamma$ and $[\gamma^{\mathrm{T}}, p^{\mathrm{T}}]=-p^{\mathrm{T}}$. Thus if the eigenvalues are the charges, the charge of the transpose is the negative of the charge of the original, which is exactly the relationship between the charge of a particle and its antiparticle. So we make the supposition that the matrix representing an antiparticle is the transpose (or adjoint) of the matrix representing the particle.

The simplest possible representation would be that each particle is represented to be a matrix with all zeros except one nonzero entry, which would be a 1. Let Z_{IJ} be the matrix with a 1 in the IJ position and zero everywhere else.

In this case, we could take the elements of the Cartan subalgebra to be the elements with their one on the diagonal.

Then, for instance, with $\gamma=Z_{II}$ as the charge generator and $p^+=Z_{IJ}$ and $p^-=Z_{JI}$, we would have

$$[\gamma,p^+]=[Z_{II},Z_{IJ}]=Z_{IJ}=p^+$$

and

$$[\gamma,p^-]=[Z_{II},Z_{JI}]=-Z_{IJ}=-p^-$$

We see that the particles on the same row as p^+ would also have ± 1 "charge," while all particles in the same column as p^- would have -1 "charge." Thus, we can align all the positively charged particles in the same row as p^+ and all the negatively charged particles on the same column as p^-.

The next question is how many particles do we try to include? Let us begin with the minimum of those electrically charged particles already mentioned: e^-, π^-, p^+, and their antiparticles.

According to the discussion above, we should align them as

$$\begin{matrix} \gamma_1 & & & e^- \\ & \gamma_2 & & \pi_- \\ & & \gamma_3 & p_- \\ e^+ & \pi^+ & p^+ & \gamma_4 \end{matrix}$$

Thus, we would have γ_4 as the generator of the electric charge. Now let us use the Lie bracket to fill in the above chart with $\pi^- = Z_{24}$ and $p^+ = Z_{43}$. Then

$$[p^+, \pi^-] = [Z_{43}, Z_{24}] = Z_{32} = n$$

Experimentally, we know that

$$(\pi^+, e^-) = \nu$$

Thus,

$$[\pi^+, e^-] = [Z_{42}, Z_{14}] = -Z_{31} = -\nu$$

The last slot is filled by observing that the hydrogen atom H is a tightly bound state of the proton and electron, $\mathrm{H} = (p^+, e^-)$. We have

$$[p^+, e^-] = [Z_{43}, Z_{14}] = -Z_{13} = -\mathrm{H}$$

Taking antiparticles as transposes, and we have the 4 by 4 matrix filled:

$$\begin{matrix} \gamma_1 & \nu & \mathrm{H} & e^- \\ \bar{\nu} & \gamma_2 & n & \pi^- \\ \bar{\mathrm{H}} & \bar{n} & \gamma_3 & p^- \\ e^+ & \pi^+ & p^+ & \gamma_4 \end{matrix} \tag{2.1}$$

However strange it may seem to have an atom in the same matrix as "elementary" particles, we must get over our prejudices, for the presence of the hydrogen atom is dictated by the requirement of closure of the Lie algebra and thus the hydrogen atom is "elementary" in the sense used by Segal.

This Lie algebra structure is "natural" in the sense that it is independent of any theory. The only hypothesis was that the interactions occurred via the Lie bracket. This hypothesis was confirmed by construction. Now come the connected problems of the incorporation of other known particles and the interpretation of this result.

Can we enlarge the matrix and include more particles by adding another row and column? If we could, there must first be a charged particle to put above the electron. As an example, let us put the muon in that position:

$$\begin{matrix} \gamma_0 & A & B & C & \mu^- \\ \bar{A} & \gamma_1 & \nu & \mathrm{H} & e^- \\ \bar{B} & \bar{\nu} & \gamma_2 & n & \pi^- \\ \bar{C} & \bar{\mathrm{H}} & \bar{n} & \gamma_3 & p^- \\ \mu^+ & e^+ & \pi^+ & p^+ & \gamma_4 \end{matrix} \tag{2.2}$$

where A, B, and C denote particles which are to be determined. If (2.2) were valid, we would have the reactions

$$A=[\mu^-, e^+]$$

$$B=[\mu^-, \pi^+]$$

$$C=[\mu^-, p^+]$$

and thus we would have the decays

$$A\Rightarrow\mu^- e^+$$

$$B\Rightarrow\mu^- \pi^+$$

$$C\Rightarrow\mu^- p^+$$

But these decay products are not observed in the decays of any known "elementary particle"; on the contrary, it is known that the μ^- and the p^+ combine to form a "mesonic atom" which resembles hydrogen. Thus, the μ^- seems to be a heavy electron. Thus, putting the muon in this position is not possible. Likewise, putting any known charged particle in this position causes similar problems.

This shows that the matrix Lie algebra cannot be enlarged to include any more particles in a way consistent with experiment. The impossibility of enlarging the matrix to include all particles shows the inadequacy of the matrix representation. We are forced to go to a representation of the Lie algebra by differential operators, i.e., as vector fields on an appropriate manifold (Abraham and Marsden, 1979). Thus, elementary particles are modeled as fX, where f is a function on the manifold underlying $U(3, 2)$ and X is an element of $U(3, 1)$ schematically represented in the above matrix. Each of the 16 entries in (2.1) then represents a family of particles having the same algebraic factor X and differing only by the function factor F.

At first glance, the representation in (2.1) appears to be very arbitrary. The major goal of this series of papers is to show that, up to isomorphism, this representation is not arbitrary, but rather carries vital information about the particles and their properties. The major difference between the representation in (2.1) and the standard picture is that all of the particles in (2.1) have been observed: neutrino, electron, proton, neutron, pion, hydrogen atom, and their antiparticles. γ_4 is the photon, while γ_2, γ_3, and γ_1 are photonlike neutral currents.

The γ_I each mediates a different interaction or force. These diagonal operators also serve as spectrum generators with eigenvalues 1, 0, or -1 (Table III). The γ_1 eigenvalue is the lepton number; thus, γ_1 is a weak neutral current. The γ_3 eigenvalue is the baryon number, and thus γ_3 is a strong neutral current. The γ_4 eigenvalue is the electric charge; thus γ_4 represents a photon. Since the sum of the four eigenvalues is zero, in a way the γ_2 eigenvalue is superfluous, but the force mediated by γ_2 seems to be a new weak force. Actually, there is not a "new" force; this analysis just shows that the interaction which has heretofore been thought of as the weak force is actually best understood as being two forces. This is not the "fifth force" which has been introduced by other researchers. But here is another reason for concluding that the matter matrix cannot be enlarged, for enlargement would require the introduction of another force.

The other surprise in (2.1) is the mixture of fermions and bosons in the same representation. Now, if the Lie algebra is to act as a representation space for the Lie group, it seems that a continuous action of $U(3, 1)$ on a particle state could take it into another state, mixing bosons and fermions, etc. But of course, this is physically impossible, so it must be mathematically impossible. Consequently, when describing the algebraic factor of the particle, we must deal only with the Lie algebra and not allow the continuous group action on the fiber bundle. This approach was suggested by Lipkin (1965): "The Lie algebra of the larger group is defined, but only a subset of the continuous transformations need have physical meaning, namely, those which produce continuous translations in space-time and not in the space of internal degrees of freedom." Perhaps even these continuous transformations are not needed if space-time itself is quantized.

Thus, following Lipkin's lead, we should expect the continuous representations of $U(3, 2)$ to enter the discussion only in the description of the function factors, i.e., only in the action of the group on the homogeneous space.

There is another way of viewing the problem: because of the origin of the Lie algebra structure as the elementary particles interacting, there cannot be a corresponding Lie group structure. The particles themselves are the operators in the Lie algebra and while one can exponentiate a matrix, how could one physically exponentiate a proton?

The γ_I eigenvalues, as roots of the Lie algebra, are preserved during interactions only if we insist that the particles interact via the Lie bracket.

Thus we have four conserved quantities (the internal quantum numbers as eigenvalues of the γ_I), but no continuous group action. This observation has an immediate theoretical consequence. Noether's theorem states that within the Lagrangian formalism there is a one-to-one correspondence between conserved quantities and the generators of continuous group actions. Thus, the Lie algebraic structure of the elementary particle interactions cannot be incorporated into the Lagrangian formalism. Instead of the Lagrangian structure of the quantum field theories we should expect the field equations to arise from the geometric and Lie group formalism.

This is a new way of using Lie algebras in particle physics. The presence of the Lie algebra does not indicate a symmetry of the system; instead, the Lie bracket models the way in which the elementary particles interact. If Noether's theorem were applicable, the only possible conserved quantity would be the particle type, which is only conserved in certain interactions; hence Noether's theorem is not applicable and we are not dealing with a Lagrangian field theory.

Since this is a radical departure from standard physical models, further remarks seem necessary. Why was the Lagrangian approach used? Schweber (1961, p. 257) states: "It is only when we consider interactions between fields that the Lagrangian approach achieves a status of its own: it is in fact the only known simple method for introducing interactions between the particles and for which a quantization procedure can be formulated." In this paper, a new type of interaction is introduced: the Lie bracket. Quantization is achieved by requiring that the vertical vectors be eigenvectors of the Cartan subalgebra as well being eigenvectors of the generalized Casimir operators of $U(3, 2)$. So it seems that this approach has passed the first test of being a viable alternative to the Lagrangian formalism.

Using the matrix representation, the interaction of a proton and electron was calculated above to yield a hydrogen atom. Using the standard operator representation ($Z_{IJ} \to u_I \partial_J$) for the same interaction, we have

$$[p^+, e^-] = [u_4\partial_3, u_1\partial_4] = -u_1\partial_3 = -\mathrm{H}$$

The negative sign does not appear to have physical significance other than to introduce a double-valued representation, $-\mathrm{H} = \mathrm{H}$; which is essential if these vertical vectors are to act as spinors. Physically, changing the order of particles must yield the same product.

Besides the interaction $p^+e^- \to \mathrm{H}$, the interaction $p^+e^- \to n\nu$ is also possible and at first glance this model seems to predict the necessity of a hydrogen atom intermediate state in the $p^+e^- \to n\nu$ interaction. But there is no need for the reaction to go this route. The kinetic energy of the p^+e^-

will create a neutrino–antineutrino pair and thus the reaction would go as follows:

$$p^+e^- \to p^+e^-\bar{\nu}\nu \to p^+[e^-, \bar{\nu}]\nu \to [p^+, [e^-, \bar{\nu}]]\nu \to 4p^+, [e^-, \bar{\nu}]]\nu \to n\nu$$

Since, as will be seen in Part II of this series, the W^- has the algebraic factor of $[\bar{\nu}, e^-]$, this seems to be a mathematical way of saying that the interaction proceeds via an exchange of the W^-.

3. GRADING THE LIE ALGEBRA

The interaction of a particle with the diagonal operators yields a multiple of the particle. The particles are eigenvectors of the diagonal operators Z_{II} with eigenvalue 1, 0, or −1. These numbers are the roots of the Lie algebra and the eigenvectors are the roots vectors.

Example:

$$[\gamma_4, e^-] = [u_4\partial_4, u_1\partial_4] = -u_1\partial_4 = -e^-$$

$$[\gamma_4, e^+] = [u_4\partial_4, u_4\partial_1] = u_4\partial_1 = e^+$$

$$[\gamma_4, H] = [u_4\partial_4, u_1\partial_3] = 0$$

The roots are 1, 0, or −1, but 1, 0, or −1 what? The units to be attached to these quantities determine the strength of the interaction. Proper scaling will allow us to adjust the interaction strengths to those observed in nature. These scaling constants will also determine the mass spectrum predicted by the model and will be discussed in a later paper in this series.

The position of the particles within the matrix is indexed by the standard numbering of the rows and columns of the matrix:

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

The particles carry a Z_2 grading defined by adding the indices $i+j$ modulo 2:

0	1	0	1
1	0	1	0
0	1	0	1
1	0	1	0

If we now form the matrix with entries $(-1)^{i+j}$, we obtain a matrix familiar from linear algebra, the matrix of the sign of the cofactor:

$$\begin{matrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{matrix}$$

This Z_2 grading agrees with the standard grading of particles with even $(=1)$ and odd $(=-1)$ half-integer spin. Furthermore, this grading is preserved under the interaction via the Lie bracket.

Since the grading is preserved under the Lie bracket, we must ask if there is an operator whose eigenvalues yield this grading. Thus we will look for a linear combination of the γ whose eigenvalues are the spins of the particles. Let $S=a\gamma_1+b\gamma_2+c\gamma_3+d\gamma_4$; then the eigenvalues associated with the particles are

$$\begin{matrix} \nu & a-b \\ \mathrm{H} & a-c \\ e^- & a-d \\ n & b-c \\ \pi^- & b-d \\ p^- & c-d \end{matrix}$$

Since we want these numbers to yield the spin of the particles, we have to solve six linear equations in three unknowns:

$$\begin{matrix} (1) & a-b=\frac{1}{2} \\ (2) & a-c=0 \\ (3) & a-d=\frac{1}{2} \\ (4) & b-c=\frac{1}{2} \\ (5) & b-d=0 \\ (6) & c-d=\frac{1}{2} \end{matrix}$$

From (2), $a=c$, while from (5) we conclude that $b=d$. Then (1), (3), and (6) are identical and incompatible with (4). If instead of (4) we take

$$b-c=-\tfrac{1}{2} \tag{4'}$$

the physics is still consistent and so are the equations. The system then reduces to $a=c$, $b=d$, and $a=b+\frac{1}{2}$. These are three equations in four unknowns and hence there is a free parameter in the solution. The general solution is then

$$a=c=\alpha+\tfrac{1}{2}$$

$$b=d=\alpha$$

Thus there are several ways of writing S:

$$S=(\gamma_1+\gamma_3)/2+\alpha(\gamma_1+\gamma_2+\gamma_3+\gamma_4)$$

Take $\alpha=0$; then $S=(\gamma_1+\gamma_3)/2$.
Take $\alpha=-\frac{1}{2}$; then $S=-(\gamma_2+\gamma_4)/2$.
Take $\alpha=-\frac{1}{4}$; then $S=(\gamma_1-\gamma_2-\gamma_3-\gamma_4)/4$.

With this information, we can solve the equation $S=-(\gamma_2+\gamma_4)/2$ for γ_2, obtaining

$$\gamma_2=-2S-\gamma_4$$

Thus we see that the γ_2 eigenvalue is not a new invariant, but just a linear combination of familiar quantum numbers, the spin and the electric charge.

There is also an explanation for the above grading in terms of Barut's (1980) suggestion that all particles are built from the proton, electron, and neutrino (the stable particles) and their antiparticles. From the results of Table I, we know that all of the particles are built from the stable particles:

$$\mathrm{H}=[p^+, e^-], \qquad \pi^-=[e^-, \bar{\nu}], \qquad n=[p^+, [e^-, \bar{\nu}]]$$

If we replace each particle in the matrix by the number of stable particles required to build the particle, we obtain

$$\begin{matrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{matrix}$$

duplicating the above even–odd pattern. This picture of elementary particles was also hinted at by Kursunoglu (1979).

Other gradings have also appeared in the literature. Gunaydin and Saclioglu (1982) dealt with a "three-dimensional graded structure." They observed that certain Lie algebras L "can be decomposed in the form: $L=L'+L^0+L^+$. Where L^0 is the Lie algebra of the maximal compact subgroup that contains an Abelian $U(1)$ factor. The generator of this $U(1)$ is simply the number operator that naturally gives us the three-dimensional graded structure." In the present model, this $U(1)$ factor could be γ_4 and the 1, 0, -1 is simply the charge of the particles represented by the generators. But the other diagonal operators also have eigenvalues of 1, 0, or -1 and also provide a "three-dimensional grading" of $u(3, 2)$. Thus, there are four such gradings and it seems better to think of a table of eigenvalues (roots of the Lie algebra) rather than a grading of the algebra.

The presence of the hydrogen atom along with the other particles one ordinarily thinks of as "elementary" comes as a shock at first. But if we are to include all particles and their interactions, this is a necessary inclusion, for the Lie algebra must necessarily be closed and since p^+ and e^- are elementary particles, their bracket $[p^+, e^-] = \mathrm{H}$ must also be in the Lie algebra and hence is elementary in some sense. But after the initial shock of including an atom among "elementary particles" wears off, we realize that this helps us to analyze the particle spectrum. Since there are different mass states of the hydrogen atom corresponding to the well-known different energy levels of the atom, the only consistent conclusion is that the higher-energy elementary particles are in the same way higher-energy levels of the 16 fundamental particles in the above matrix. These higher energy states are identified in Part II of this series. Other "elementary particles" are analogues of nuclei formed from these excitations.

Dyson (1966) noted "a growing similarity, both in the observed data and in the theoretical analysis, between nuclear and particle physics. This point of view may be regarded by some people as nihilistic, so far as particle physics is concerned, since it implies that the continued discovery of more and more resonances may be ultimately as unilluminating as the discovery of more and more isometric states of nuclei." This "growing similarity" also extends to the high Rydberg states of atoms. While interesting in their own right, the discovery of new excited states of particles cannot induce the same excited states in physicists as they once did.

This classification of the elementary particles is a radical departure from the Gell-Mann classification. Gell-Mann placed particles of the same mass into a representation or multiplet. Here, the complexification of $u(3, 1)$ includes the photon and the neutrino as well as the proton and hydrogen atom, from massless to very massive. The success of Gell-Mann's multiplet program indicates that the Casimir operator of $SU(3)$ is included in the mass operator. Since $SU(3)$ is a subgroup of $SU(3, 1)$, we assure this by taking the Casimir operator of $U(3, 2)$ as the energy operator.

This identification of particles with the Lie algebra of $u(3, 1)$ is novel in several respects. Usually when dealing with noncompact Lie algebras, physicists have insisted that only the compact generators can represent particles. The reason elementary particle physicists eschewed noncompact groups is that the unitary representation of noncompact groups is infinite dimensional and in the Gell-Mann classification this would lead to the absurd conclusion that there is an infinite number of elementary particles of the same mass. There is an infinite number of high-energy states of the hydrogen atom, although the higher levels are very unstable and difficult to produce. In the above list, the noncompact generators represent the negative norm states as well as the charged particles. Bracketing with a noncompact generator changes the particle type from compact (neutral) to noncompact (charged) and vice versa. This phenomenon occurs in the Dirac theory, where "the electric field induces transitions of the particle between the positive- and negative-energy states of a free particle" (Preparata, 1979, p. 33). The big difference is that in the present picture, the electric field is not a perturbation, but is built into the geometry of the particle and furthermore, it is the interaction with the charged particle itself, not just the electric field, which induces the transition from positively normed states to negatively normed states. Thus, except for interaction with a charged particle, there is no transition permitted between positively normed states and negatively normed states.

For each of the 16 elements or the basis of the Lie algebra $u(3, 1)$ there is a family of particles. Thus, for each of these basis elements there is a space of functions which can act as the function factors of that element of the Lie algebra. Essentially, then, there are 16 "Hilbert spaces," one for each basis element of the Lie algebra $u(3, 1)$. In the standard Fock space approach to quantum field theory, these 16 spaces would be put together as a tensor product. Likewise, in the present picture, there will be an infinite number of particles or resonances within each of the 16 families of particles as well as many other particles formed like nuclei from these excited states and modeled as tensor products. Here, the total space is not just a tensor product, but carries a Lie algebra structure. Then there are many more particles to be discovered within the present framework. The theory does predict that all particles fall into this framework.

Since the only consistent interpretation of $Z_{13} = [p^+, e^-]$ is as the hydrogen atom, the distinction between particle physics and atomic physics is blurred from the outset. The line between particle physics and nuclear physics is just as unclear, since, as we will see, several of the "elementary particles" are composite.

The list of "fundamental particles" contains all of the "everyday" particles: the pre-1935 particles. The appearance of the hydrogen atom on this list forces us to conclude that the other "elementary particles" are excitations or combinations of the fundamental particles in the same way that high Rydberg states are excited states of the hydrogen atom and nuclei are combinations of protons and neutrons. These excitations will have a different function factor and the combinations allowed will be modeled as tensor products. We must expect the fundamental particles to be the lowest energy states for the given algebraic factor.

4. THE GEOMETRY OF ELEMENTARY PARTICLE INTERACTIONS

To model a particle as fX requires an intimate relation between the function factor f and the algebraic factor X. This relation will be the subject of a later paper in this series; suffice it for now to say that f will be an eigenfunction of several operators—the generalized Casimir operators of $U(3, 2)$—and associated with each X is a dynamical group chain from which these operators will be constructed. These operators determine the mass, momentum, and other invariants of the particle.

This relation between f and X is essentially a relation between the geometry of the base and the geometry of the particle interactions which take place in the vertical bundle. We will show that it is possible to interpret these vertical vectors in terms of the curvature of the base. The space-time indices of $u(3, 2)$ were suppressed in the discussion of particles. The full parameterization of the complexification of $u(3, 2)$ would be

$$\begin{matrix} \gamma_1 & \nu & \mathrm{H} & e^- & Q_1 \\ \bar{\nu} & \gamma_2 & n & \pi^- & Q_2 \\ \bar{\mathrm{H}} & \bar{n} & \gamma_3 & p_- & Q_3 \\ e^+ & \pi^+ & p^+ & \gamma_4 & Q_4 \\ P_1 & P_2 & P_3 & P_4 & \gamma_5 \end{matrix} \tag{4.1}$$

where the generators of $U(3, 2)$, i.e., $X_1 = Q_1 + P_1$, $X_2 = Q_2 + P_2$, $X_3 = Q_3 + P_3$, $X_4 = Q_4 - P_4$, $Y_1 = i(Q_1 - P_1)$, $Y_2 = i(Q_2 - P_2)$, $Y_3 = i(Q_3 - P_3)$, and $Y_4 = i(Q_4 + P_4)$; are the basis for the tangent space of the complex space-time QAdS. The X_I are the tangent vectors of AdS.

The bracket of Q_I with a conjugate momentum P_J yields a "particle." This is illustrated in Table III. Technically, these vertical vectors, or particles, arise as the sectional curvature of QAdS. There are no quarks, but rather, the dimensions of QAdS replace quarks as the building blocks of matter, but not just hadronic matter, for in this geometry, the leptons are built from the same "stuff": space-time itself. The basic building blocks of matter are not particles, but the coordinates of QAdS, i.e., the dimensions of (complex) space-time itself. Bracket them and one obtains a particle. There are no quarks, unless the coordinates of space-time are considered to be the quarks (Preparata, 1979).

Now, since the X_I and Y_J are the real tangent vectors, we should be looking at the brackets $[X_I, Y_J]$ instead of the brackets $[Q_I, P_J]$. Let us then compute

$$\begin{aligned}[X_4, Y_1] &= [Q_4 - P_4, i(Q_1 - P_1)] \\ &= [Q_4, i(Q_1 - P_1)] - [P_4, i(Q_1 - P_1)] \\ &= [Q_4, iQ_1)] - [Q_4, iP_1)] - [P_4, iQ_1] + [P_4, iP_1] \\ &= -i[Q_4, P_1)] - i[P_4, Q_1] \\ &= -i(e^+ + e^-)\end{aligned}$$

Thus, when we look at the bracket of the real tangent vectors, we see that to produce a particle, we automatically obtain the antiparticle! There is truly some geometric magic going on here!

The new diagonal term, γ_5, in the full representation is evidently the graviton. The graviton then commutes with the algebraic factor of all the particles. Thus, the graviton acts only on the function factor of the particles. This implies that the gravitational interaction is fundamentally different from the other interactions and since all particles must have a function factor, this allows us to conclude that all particles must interact via gravitation.

Using the operator representation, $Q_I = u_I \partial_5$, $P_I = u_5 \partial_I$, $\gamma_5 = iu_5\partial_5$, we see that $[\gamma_5, Q_I] = [iu_5\partial_5, u_I\partial_5] = -iu_I\partial_5 = -iQ_I$ and $[\gamma_5, P_I] = [iu_5\partial_5, u_5\partial_I] = iu_5\partial_I = iP_I$. Thus

$$\begin{aligned}[\gamma_5, X_I] &= [\gamma_5, Q_I + P_I] = -i(Q_I - P_I) = -Y_I \\ [\gamma_5, Y_I] &= [\gamma_5, Q_I - P_I] = (Q_I + P_I) = X_I\end{aligned}$$

The complex tangent spaces to QAdS are the eigenvectors corresponding to the eigenvalue of $+i$:

$$T^{(1,0)}\text{QAdS} = \text{linear span of } Q_I$$

and the eigenvectors corresponding to the eigenvalue of $-I$:

$$T^{(0,1)}\text{QAdS} = \text{linear span of } P_I$$

This calculation shows that γ_5 defines the complex structure on QAdS by $JX = [\gamma_5, X]$ and in turn the space of complex structures on space-time is the space of twistors. Thus, there will prove to be some very interesting connections between the present work and the twistor program.

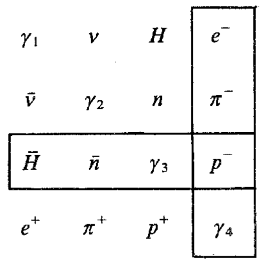

Fig. 1. The proton may interact via the bracket with any of the boxed particles to change particle type.

The rules for interaction via the Lie bracket (the interactions in which a change in particle type occurs) lead to the following rules for determining the results of the interaction without computing any brackets:

1. A particle may interact with its antiparticle or with any particle in the same row or column as the antiparticle (Figure 1).
2. The two interacting particles determine two corners of a rectangle with the third corner on the diagonal.
3. The results of the interaction are the two particles on the opposite corners (one of which must be a γ_I by rule 2).

Figure 2 illustrates a possible interaction. The others are listed in Table I. For simplicity, the negative signs have been suppressed. All other particles are excited states of the above particles, much like high Rydberg states of the hydrogen atom, or they are modeled as tensor products of the above and resemble nuclei more than elementary particles. These higher energy states will be discussed in Part II of this series of papers.

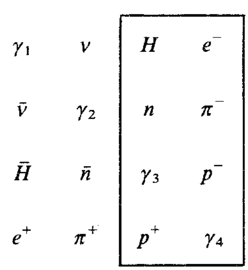

Fig. 2. $H = [p^+, e^-]$.

Table I. The Interactions Predicted by the Lie Bracket

$[p^+, p^-] = \gamma_3 - \gamma_4$	$[\pi^-, \pi^+] = \gamma_2 - \gamma_4$
$[e^+, e^-] = \gamma_1 - \gamma_4$	$[n, \bar{n}] = \gamma_2 - \gamma_3$
$[\nu, \bar{\nu}] = \gamma_1 - \gamma_2$	$[\mathrm{H}, \bar{\mathrm{H}}] = \gamma_3 - \gamma_1$
$n = [p^+, \pi^-] = [p^+, [e^-, \bar{\nu}]]$	$[p^-, \pi^+] = \bar{n}$
$H = [p^+, e^-]$	$[e^+, p^-] = \bar{\mathrm{H}}$
$[\bar{H}, p^+] = e^+$	$[\mathrm{H}, p^-] = e^-$
$[e^+, \pi^-] = \bar{\nu}$	$[e^-, \pi^+] = \nu$
$[\pi^+, n] = p^+$	$[\bar{n}, \pi^-] = p^-$
$[\pi^+, \bar{\nu}] = e^+$	$[\nu, \pi^-] = e^-$
$[e^+, \mathrm{H}] = p^+$	$[\bar{\mathrm{H}}, e^-] = p^-$
$[e^+, \nu] = \pi^+$	$[\bar{\nu}, e^-] = \pi^-$
$[\bar{\mathrm{H}}, n] = \bar{\nu}$	$[\mathrm{H}, \bar{n}] = \nu$
$[\bar{\mathrm{H}}, \nu]\bar{n}$	$[\bar{\nu}, \mathrm{H}] = n$
$[\bar{\nu}, \bar{n}] = \bar{\mathrm{H}}$	$[\nu, n] = \mathrm{H}$
$[n, p^-] = \pi^-$	$[p^+, \bar{n}] = \pi^+$

Once the above rules for interactions are understood, there is an obvious generalization possible:

1. Take any two particles not in the same row or column.
2. Form the rectangle determined by these particles.
3. The results of the interaction of the given particles are the particles on the opposite corners. This interaction can go either way, depending on the relative energies.

There are only six possibilities generated by the generalized rules not listed in Table I; these will be called secondary interactions and are listed in Table II.

These secondary interactions cannot be described in terms of the bracket in the matrix representation. But the secondary interactions can be described in terms of the curvature of $U(3, 2)$.

The general curvature 4-tensor on a homogeneous space is given by

$$R(X, Y, U, V) = B(R(X, Y)U, V)$$

Table II. The Secondary Interactions

$\mathrm{H}\pi^- \leftrightarrow ne^-$	$\bar{\mathrm{H}}\pi + \leftrightarrow \bar{n}e^+$
$\mathrm{H}\pi^+ \leftrightarrow p^+\nu$	$\bar{\mathrm{H}}\pi^- \leftrightarrow p^-\bar{\nu}$
$ne^+ \leftrightarrow p^+\bar{\nu}$	$\bar{n}e^- \leftrightarrow p^-\nu$

where B is the Killing form on $u(3, 2)$. By elementary properties of the Killing form and using

$$R(X, Y)U = -[[X, Y], U]$$

which is valid for a reductive homogeneous space, we obtain

$$\begin{aligned} R(X, Y, U, V) &= B(R(X, Y)U, V \\ &= -B([[X, Y], U], V) \\ &= -B([X, Y], [U, V]) \end{aligned}$$

Now let us consider one of the secondary interactions:

$$H\pi^- \leftrightarrow ne^-$$

From Table III we obtain

$$\begin{aligned} \mathrm{H} &= [P_3, Q_1] \\ \pi^- &= [P_4, Q_2] \\ n &= [P_3, Q_2] \\ e^- &= [P_4, Q_1] \end{aligned}$$

Table III. Particles as the Curvature of QAdS

Particle	Algebraic factor	Spacetime bracket	Charges γ_1	γ_2	γ_3	γ_4
ν	$u_1\partial_2$	$[Q_1, P_2]$	1	−1	0	0
$\bar{\nu}$	$u_2\partial_1$	$[Q_2, P_1]$	−1	1	0	0
H	$u_1\partial_3$	$[Q_1, P_3]$	1	0	−1	0
$\bar{\mathrm{H}}$	$u_3\partial_1$	$[Q_3, P_1]$	−1	0	1	0
e^-	$u_1\partial_4$	$[Q_1, P_4]$	1	0	0	−1
e^+	$u_4\partial_1$	$[Q_4, P_1]$	−1	0	0	1
n	$u_2\partial_3$	$[Q_2, P_3]$	0	1	−1	0
$\bar{n}$	$u_3\partial_2$	$[Q_3, P_2]$	0	−1	1	0
π^-	$u_2\partial_4$	$[Q_2, P_4]$	0	1	0	−1
π^+	$u_4\partial_2$	$[Q_4, P_2]$	0	−1	0	1
p^-	$u_3\partial_4$	$[Q_3, P_4]$	0	0	1	−1
p^+	$u_4\partial_3$	$[Q_4, P_3]$	0	0	−1	1
γ_1	$u_1\partial_1 - u_5\partial_5$	$[Q_1, P_1]$	0	0	0	0
γ_2	$u_2\partial_2 - u_5\partial_5$	$[Q_2, P_2]$	0	0	0	0
γ_3	$u_3\partial_3 - u_5\partial_5$	$[Q_3, P_3]$	0	0	0	0
γ_4	$u_4\partial_4 - u_5\partial_5$	$[Q_4, P_4]$	0	0	0	0

Thus, the interaction of H with π^- via curvature might be written as

$$\begin{aligned} B([P_3, Q_1], [P_4, Q_2]) &= -B(Q_1, [P_3[P_4, Q_2]]) \\ &= B(Q_1, [[P_3, P_4], Q_2] + [P_4, [P_3, Q_2]]) \\ &= B(Q_1, [P_4, [P_3, Q_2]]) \\ &= -B([P_4, Q_1], [P_3, Q_2]) \end{aligned}$$

Beginning with $H\pi^-$ and following the rules for calculating the curvature, we arrived at ne^-. Thus, the change in particle type may be interpreted as an interaction which preserves curvature. In the same way, all of the above interactions may be interpreted as interactions involving curvature. Thus, it seems that particles can be interpreted as the curvature of space-time and the dynamics required is the dynamics of curvature, exactly as Einstein envisioned. Except now, the curvature involved is that of $U(3, 2)$, not just space-time, and there are five interactions involved, not just gravitation. Just as Lurcat (1964) used the group manifold of the Poincaré group to obtain a dynamical role of spin, so here we use the group manifold of $U(3, 2)$ to obtain the dynamics of a totally unified field theory. Also, this calculation shows that the dynamics of the full curvature tensor is involved, not just the trace, so the present theory is not just a generalization of the Einstein theory to higher dimensions.

We can now count the different types of interactions in terms of the "interaction rectangles."

Theorem. There are 36 interaction rectangles.

Proof. There are ${}_4C_2 = 6$ ways to choose the two columns defining the left and right sides of the rectangle and ${}_4C_2 = 6$ ways to choose the top and bottom. Thus there are $6 \times 6 = 36$ possible rectangles, all of which are listed in Tables I and II. ■

The listing of these interactions takes us as far as the purely algebraic approach can go. The full description of these particles requires that the Lie algebra be represented as differential operators, i.e., as vector fields on QAdS. In this more complete picture, the above matrices will be replaced by vertical vector fields (differential operators) $X, Y \in u(3, 1) \times u(1)$ and the particles will be modeled as fX, hY with the standard interaction of vector fields:

$$[fX, hY] = f(Xh)Y - h(Yf)X + fh[X, Y] \tag{4.2}$$

This mode of particle interaction is reminiscent of the theory of currents, except that in that formulation the first two terms are not present. Thus, in the theory of currents, instead of (4.2) we would have (Herman, 1972)

$$[fX, hY] = fh[X, Y] \tag{4.3}$$

But the "extra terms" in (4.2) are necessary to allow for the probability that the two particles interact, but not in a way which would change particle type. Also, taking the interaction of elementary particles as (4.3), we would lose the geometric meaning of the interaction as the Lie bracket of vector fields.

Gravitation is different from the other forces in that the mass does not appear to be an additive quantum number, and hence is not a root of the Lie algebra. Consider the interaction of the proton with its antiparticle, $[p^+, p^-] = \gamma_3 - \gamma_4$. The resulting quanta are massless. For mass to be an additive quantum number would require that the mass of the antiproton be the negative of the mass of the proton. The meaning of negative mass is not clear to this author. Since mass is not an additive quantum number, it is not an eigenvalue of a first-order differential operator and should then perhaps be related to the eigenvalue of the second-order Casimir operator of $u(3, 2)$. Indeed, we will see that the eigenvalue of the second-order Casimir operator of $u(3, 2)$ is the total energy, the eigenvalue of the second-order Casimir operator of $u(3, 1)$ is the internal energy, and the difference of these two eigenvalues is the Laplace–Beltrami operator on QAdS and its eigenvalue is the relativistic mass.

5. QUESTIONING THE CONNECTION BETWEEN SPIN AND STATISTICS

The study of the connection between spin and statistics has a long and venerable history, as related by Pais (1986), and the topic is now standard fare in treatments of quantum field theory. The spin–statistics theorem states that particles with integer spin obey Bose–Einstein statistics, while particles with half-integer spin obey Fermi–Dirac statistics. The exact hypotheses under which the theorem is true evolved from the 1930s and culminated with Burgoyne (1958) showing that the theorem is true in any field theory satisfying the standard hypotheses:

1. Invariance under proper, orthochronous, Poincaré transformations.
2. There are no negative-energy states.
3. The metric on Hilbert space is positive definite.
4. At spacelike separations, distinct fields either commute or anticommute.

In the present work:

1. We replace the Poincaré group by $U(3, 2)$.
2. We replace the standard Hilbert space of functions by the space of vertical vectors on $U(3, 2)/U(3, 1) \times U(1)$.
3. The natural "metric" on this space is not positive definite and negative-energy states may exist.

Thus, the first three of Burgoyne's hypotheses are not satisfied. Can the spin–statistics theorem be proven in this context? A negative answer seems inevitable. But if the theorem cannot be proven from the model, the obvious question must be, Is the theorem true? Within the model of matter introduced here, there are two families of particles which seem to cause problems with the spin–statistics connection.

The hydrogen atom consists of a proton and an electron. Since the proton and the electron are both fermions, the hydrogen atom has integer spin and should be a boson. But particles obeying Bose–Einstein statistics do not satisfy the Pauli exclusion principle and a large number of bosons can be gathered in an arbitrarily small region of space-time. If we put a large number of hydrogen atoms into a small space, there is an imposed structure as required by the Pauli exclusion principle. For by combining hydrogen atoms we obtain hydrogen molecules. The spectrum of these hydrogen molecules is the direct result of the hydrogen atoms obeying the Pauli exclusion principle. We must conclude that hydrogen atoms do not obey Bose statistics.

A quite different argument holds for a collection of charged pions. The pion consists of an electron and an antineutrino and thus has integral spin, implying that it should obey Bose statistics, and hence, any number of π^- can occupy the same space. Since they are charged, the laws of electrodynamics imply that to put two π^- into the same space would require an infinite amount of energy. Thus two basic principles of quantum field theory, the spin–statistics theorem and electromagnetic repulsion of two particles with the same charge, are in conflict. Can a large number of π^- occupy the same space? Since they have integer spin, the spin–statistics theorem says yes, but since they have the same charge, electrostatics says no. I believe that the more basis of these two "laws" must hold and that the electrostatic repulsion will win out. Of course, this conjecture needs to be experimentally tested.

Electrons obey the Pauli exclusion principle. The direct result of this is a structure of atoms with different electrons in different states. If the π^- obeys the Pauli exclusion principle, there should be a corresponding structure and indeed there is. In the picture developing here, nuclei consist of protons exchanging π^-. Thus, the structure of nuclei is due to the pions and the protons obeying the Pauli exclusion principle. We must conclude that charged pions do not obey Bose statistics.

I then suggest that there is no connection between spin and statistics and that only the diagonal particles actually obey Bose statistics. This satisfies the need to have bosons represented by commuting operators. The "diagonal particles" are represented by the Cartan subalgebra of $u(3, 1) \times u(1)$, which is precisely the maximal Abelian subalgebra. The fermions would then be represented by the noncommuting elements of the algebra.

6. CONCLUSION

In this paper I have introduced a new classification of the first generation of elementary particles and modeled their interactions via the Lie bracket. In Part II of this series the "excited states" of these particles are identified. Identification of these excited states is a necessary prelude to an explicit calculation of the masses of these particles.

In the geometric framework presented here, there are no Higgs particles, no axions, no monopoles—so far, only the observed first-generation particles. There is no need for renormalization: since all of the numbers have a geometric meaning, they are finite from the start. The spin, charge, baryon number, and lepton number are among these geometric numbers (roots of the Lie algebra). They are strictly conserved and hence there is no proton decay. The parities arise from the geometry without the need for a "superalgebra." Since the particles are realized as part of the geometry of complex space-time, we have hope that this may lead to a complete fusion of quantum physics and gravity (Kaiser, 1990; Charon, 1988).

Quantization via the generalized Casimir operators essentially "reduces" quantum mechanics to harmonic analysis on the space $U(3, 2)/U(3, 1) \times U(1)$, a relation which has been hinted at before (Rawnsley *et al.*, 1983).

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